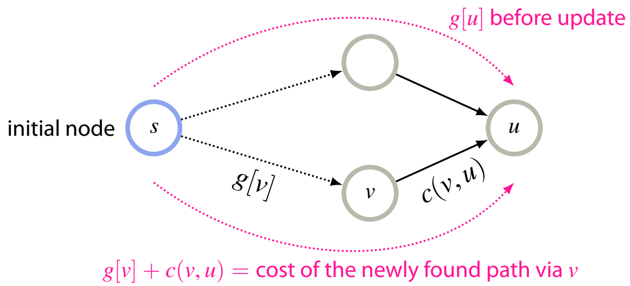
"Relaxing" operation for edge (v, u)

Maintaining the best path from the initial node



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if g[v] + c(v,u) < g[u] then
    g[u] ← g[v] + c(v,u)          # Update g[u] if a shorter path is found
    Parent[u] ← v                 # Update Parent[u] to the parent node v
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Property of $g[u]$ and the "relax" operation

Let $g^*(u)$ denote the cost of the shortest path to node u.

Upper-boundedness

For any node u, $g[u] \geq g^*(u)$.
Once we have $g[u] = g^*(u)$, $g[u]$ does not change anymore.

Convergence

If $s \rightsquigarrow v \to u$ is a shortest path from s to u (and thus $s \rightsquigarrow v$ is a shortest path from s to v), and if $g[v] = g^*(v)$, after relaxing (v, u), it holds that $g[u] = g^*(u)$.

Theorem

At any moment during a run of Dijkstra's shortest path algorithm, for any node v in CLOSED,

$$g[v] = g^*(v)$$

Proof of the theorem:

At any moment during a run of Dijkstra's shortest path algorithm, the following invariant holds.

For any node v in CLOSED,

$$g[v] = g^*(v)$$

- In the beginnning of the algorithm, CLOSED is empty. Hence the statement holds.
- Because $g[s] = g^*(s) = 0$, when s enters CLOSED, the statement still holds.

Proof after s has entered CLOSED

We use the "proof by contradiction" technique

Assume on the contrary that

> "at some point during a run of Dijkstra's algorithm, a node u such that $g[u] > g^*(u)$ enters CLOSED"

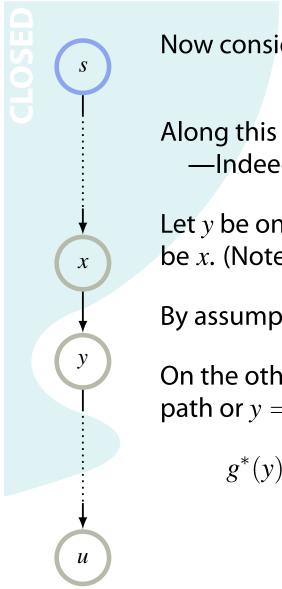
and

show that this leads to a contradiction.

Let u be the first such node during the run, and consider the moment immediately before u enters CLOSED (That is, u is on OPEN and has not entered CLOSED yet)

Thus, at this moment, for all v in CLOSED, we have

$$g[v] = g^*(v)$$



Now consider a shortest path from s to u.

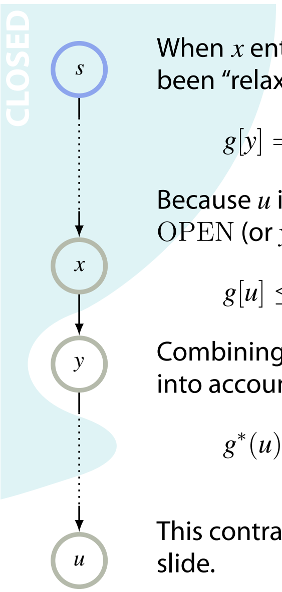
Along this path, there exists at least one node not placed in CLOSED.
—Indeed, u is not in CLOSED (yet)

Let y be one such node closest to s along this path, and let its parent be x. (Note: It may be that $x = s$ or $y = u$)

By assumption, $x \in \mathrm{CLOSED}$. Hence $g[x] = g^*(x)$

On the other hand, because either y precedes u along this shortest path or $y = u$, we have

$$g^*(y) \le g^*(u)$$



When x entered CLOSED with $g[x] = g^*(x)$, edge (x, y) must have been "relaxed." It follows that

$$g[y] = g^*(x) + c(x, y) = g^*(y) \qquad (1)$$

Because u is on OPEN and about to be expanded, and y is also on OPEN (or $y = u$), we have

$$g[u] \le g[y] \qquad (2)$$

Combining Eqs. (1) and (2), and taking the assumption $g^*(u) < g[u]$ into account,

$$g^*(u) < g[u] \le g[y] = g^*(y)$$

This contradicts the inequality $g^*(y) \le g^*(u)$ shown in the previous slide.

Thus, we have established the theorem:

At any moment during a run of Dijkstra's algorithm, for any node v in CLOSED, we have

$$g[v] = g^*(v).$$

(That is, for all nodes in CLOSED, we have already found a shortest path to them.)

No need to do anything when a successor node v is already closed.

Moreover, when a goal node $t \in G$ is closed, it has $g[t] = g^*(t)$.